PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 18, Fasc. 2 (1998), pp. 351–357

SOME REMARKS ON J_0-REGULARITY AND J_0-SINGULARITY OF q-VARIATE STATIONARY PROCESSES

BY

LUTZ KLOTZ (LEIPZIG) AND FRANZ SCHMIDT (DRESDEN)

Abstract. We give a new proof of Makagon's and Weron's criterion for J_0-regularity (see [4], Theorem 5.3), and discuss some conditions of J_0-singularity of q-variate stationary processes.

1. The present short paper is devoted to the study of J_0-regularity and J_0-singularity of a q-variate stationary process on a discrete Abelian group G, where the family J_0 consists of the complements of singletons of G. (For detailed definitions of the notions concerning q-variate stationary processes we refer to [3] or [4].) We give a certain description of the space which bears the J_0-singular part of the process and use it to derive a criterion of J_0-regularity proved by Makagon and Weron ([4], Theorem 5.3). We treat in this paper only finite-dimensional stationary processes. For results concerning infinite-dimensional stationary sequences the reader may consult [2]. Applying a method due to Matveev [5] we obtain some conditions necessary or sufficient for J_0-singularity.

2. Let N, Z and C be the set of positive integers, the Abelian group of integers, and the field of complex numbers, respectively. For $q \in N$, denote by C^q the q-dimensional Hilbert space of complex column vectors of length q, and by M_q the linear space of all $(q \times q)$-matrices with complex entries. The inner product in C^q is denoted by (u, v), $u, v \in C^q$. The elements of M_q will often be interpreted as linear operators on C^q. For a subspace K of C^q, denote by $K^\perp$ its orthogonal complement, and by P_K the orthoprojector onto K. For $A \in M_q$, the symbols $\operatorname{Ker}(A)$, $R(A)$, $\det(A)$, A^*, and $A^{\#}$ stand for the kernel, the range, the determinant, the adjoint, and the Moore–Penrose inverse of A, respectively.

Let G be any discrete Abelian group, Γ its dual, and σ the Haar measure on Γ. Throughout the paper, relations between (Borel) measurable functions on Γ are to be understood as relations which hold σ-almost everywhere (abbreviated to σ-a.e.). In all integrals the domain of integration will be Γ. The value of $\gamma \in \Gamma$ at $g \in G$ will be denoted by $\langle g, \gamma \rangle$.

Let $X := (X_g)_{g\in G}$ be a q-variate stationary process over G, F its spectral measure, and $L^2(F)$ its spectral space. For $g \in G$, set

$$\mathscr{W}_g := \bigvee \{\langle h, \cdot\rangle \colon h\in G,\ h \neq g\},$$

where the symbol $\bigvee$ denotes the closed matrix linear hull in $L^2(F)$. The process X is called J_0-*regular* if $\mathscr{W}_\infty := \bigcap_{g\in G} \mathscr{W}_g = \{0\}$ and J_0-*singular* if $\mathscr{W}_\infty = L^2(F)$.

Let $dF = W d\sigma + dF_s$ be the Lebesgue decomposition of F, and $X = Y+Z$ the corresponding decomposition of X. Then the process X is J_0-regular if and only if Y is J_0-regular and $Z = 0$. It is J_0-singular if and only if Y is J_0-singular. This shows that if we want to study J_0-regularity or J_0-singularity, we can confine ourselves to the case where the spectral measure is absolutely continuous (with respect to σ). We will do so and assume throughout the paper that the process X has a spectral measure of the form $dF = W d\sigma$. Then the function W will be called a *spectral density matrix* and the spectral domain will be denoted by $L^2(W)$. The space $L^2(W)$ is the Hilbert space of (equivalence classes of) measurable M_q-valued functions Φ such that the integral $\int \Phi W \Phi^* \, d\sigma$ exists.

It is not hard to see that the map

$$U\colon \Phi \to \Phi W, \qquad \Phi \in L^2(W),$$

is an isometric isomorphism from $L^2(W)$ onto $L^2(W^{\#})$.

For $g\in G$, let $\mathscr{V}_g$ be the (matrix) orthogonal complement of $\mathscr{W}_g$ in $L^2(W)$. The following lemma is the basis of our investigations.

LEMMA 1 (cf. [3], Theorem 3.4 and Lemma 3.7). *For $g\in G$, the set $U\mathscr{V}_g$ consists of all M_q-valued functions $\langle g, \cdot\rangle A$ such that*

(i) $A \in M_q$ *and* $\operatorname{Ker}(A) \supseteq \operatorname{Ker}(W)$ *σ-a.e.*;

(ii) $\langle g, \cdot\rangle A \in L^2(W^{\#})$.

3. Now we examine conditions (i) and (ii) of the preceding lemma separately. We start with condition (ii).

Let $\mathscr{C}$ be the set of all constant M_q-valued functions belonging to $L^2(W^{\#})$. Of course, we can and will identify $\mathscr{C}$ with a subset of M_q.

LEMMA 2. *If $A\in\mathscr{C}$, $B \in M_q$, and $\operatorname{Ker}(B) \supseteq \operatorname{Ker}(A)$, then $B\in\mathscr{C}$.*

Proof. If $\operatorname{Ker}(B) \supseteq \operatorname{Ker}(A)$, then $BA^{\#}A = B$, and the lemma follows from the fact that $L^2(W^{\#})$ is a left M_q-module.

LEMMA 3 (cf. [3], Lemma 4.3). *If $A\in M_q$, then $A\in\mathscr{C}$ if and only if $P_{R(A^*)} \in \mathscr{C}$.*

Proof. Since $\operatorname{Ker}(A) = \operatorname{Ker}(P_{R(A^*)})$, Lemma 3 is an immediate consequence of Lemma 2.

LEMMA 4. *Let $n\in N$ and $A_j \in \mathscr{C}$, $j = 1, \ldots, n$. If*

$$B\in M_q \quad \text{and} \quad \operatorname{Ker}(B) \supseteq \bigcap_{j=1}^{n} \operatorname{Ker}(A_j),$$

then $B\in\mathscr{C}$.

Proof. Since $A_j \in \mathscr{C}$, from Lemma 3 we obtain $P_{R(A_j^*)} \in \mathscr{C}$, $j = 1, \ldots, n$. Hence $A := \sum_{j=1}^{n} P_{R(A_j^*)}$ belongs to $\mathscr{C}$. Let $u \in \operatorname{Ker}(A)$. Then

$$\left(\sum_{j=1}^{n-1} P_{R(A_j^*)} u, u\right) = -(P_{R(A_n^*)} u, u),$$

which yields

$$u \in \operatorname{Ker}(P_{R(A_n^*)}) \quad \text{and} \quad u \in \operatorname{Ker}\left(\sum_{j=1}^{n-1} P_{R(A_j^*)}\right).$$

Repeating the argument, we eventually obtain $u \in \operatorname{Ker}(P_{R(A_j^*)}) = \operatorname{Ker}(A_j)$, $j = 1, \ldots, n$. Since the inclusion $\bigcap_{j=1}^{n} \operatorname{Ker}(A_j) \subseteq \operatorname{Ker}(A)$ is obvious, we get

$$\operatorname{Ker}(A) = \bigcap_{j=1}^{n} \operatorname{Ker}(A_j).$$

But A belongs to $\mathscr{C}$. Thus an application of Lemma 2 completes the proof.

Let $\mathscr{A}_1 := \{A \in M_q \colon A \in \mathscr{C} \text{ and } A^* \in \mathscr{C}\}$.

LEMMA 5. *Let $A \in M_q$. Then $A \in \mathscr{C}$ if and only if $P_{R(A^*)} \in \mathscr{A}_1$.*

Proof. Use Lemma 3 and the fact that $P_{R(A^*)} \in \mathscr{C}$ if and only if $P_{R(A^*)} \in \mathscr{A}_1$.

LEMMA 6. *The set $\mathscr{A}_1$ is a von Neumann algebra.*

Proof. Clearly, $\mathscr{A}_1$ is a *-algebra. It remains to show that $\mathscr{A}_1$ is closed. Let $\{A_n\}_{n \in N} \subseteq \mathscr{A}_1$ be a sequence converging to $A \in M_q$ (with respect to the topology of M_q). Since C^q has finite dimension, there exists an $n_0 \in N$ such that

$$\bigcap_{n=1}^{n_0} \operatorname{Ker}(A_n) = \bigcap_{n=1}^{\infty} \operatorname{Ker}(A_n).$$

Since $\operatorname{Ker}(A) \supseteq \bigcap_{n=1}^{\infty} \operatorname{Ker}(A_n)$, Lemma 4 yields $A \in \mathscr{C}$. In an analogous way one obtains $A^* \in \mathscr{C}$.

LEMMA 7. *There exists a subspace H_1 of C^q such that a $(q \times q)$-matrix A belongs to $\mathscr{C}$ if and only if* $\operatorname{Ker}(A) \supseteq H_1$.

Proof. According to Proposition 5.1.8 in [1] the von Neumann algebra $\mathscr{A}_1$ contains a maximal orthoprojector, i.e. an orthoprojector P_1 such that $R(P_1) \supseteq R(A)$ for all $A \in \mathscr{A}_1$. Let $H_1 := \operatorname{Ker}(P_1)$. Then $H_1 \subseteq \operatorname{Ker}(A^*)$, and hence $H_1 \subseteq \operatorname{Ker}(A)$ for all $A \in \mathscr{A}_1$, and Lemma 5 yields $H_1 \subseteq \operatorname{Ker}(A)$ for all $A \in \mathscr{C}$. On the other hand, since $P_1 \in \mathscr{C}$, Lemma 2 implies that $A \in \mathscr{C}$ for all $A \in M_q$ such that $\operatorname{Ker}(A) \supseteq H_1$.

The next lemma sheds some light on condition (i) of Lemma 1.

LEMMA 8. *There exists a subspace H_2 of C^q with the following properties:*

(1) $\operatorname{Ker}(W) \subseteq H_2$ *σ-a.e.*

(2) *If for any subspace K of C^q the relation* $\operatorname{Ker}(W) \subseteq K$ *σ-a.e. holds, then* $H_2 \subseteq K$.

Proof. Set $\mathscr{A}_2 := \{A \in M_q\colon AP_{R(W)} = P_{R(W)}A = A \ \sigma\text{-a.e.}\}$. Clearly, $\mathscr{A}_2$ is a von Neumann algebra, and an orthoprojector $Q \in M_q$ belongs to $\mathscr{A}_2$ if and only if $\mathrm{Ker}(Q) \supseteq \mathrm{Ker}(W)$ σ-a.e. Using these facts it is not hard to see that if P_2 denotes the maximal orthoprojector of $\mathscr{A}_2$, the space $H_2 := \mathrm{Ker}(P_2)$ has properties (1) and (2).

Now it is easy to obtain a certain description of the set $U\mathscr{V}_g$.

LEMMA 9. *There exists a subspace H of C^q such that*

$$U\mathscr{V}_g = \{\langle g, \cdot\rangle A\colon A \in M_q, \mathrm{Ker}(A) \supseteq H\}, \qquad g \in G.$$

Proof. Combining the results of Lemmas 1, 7 and 8, we see that the space $H := H_1 + H_2$ has the desired property.

In the rest of the paper, we will denote by H_1, H_2 and H the subspaces of Lemmas 7, 8 and 9, respectively.

PROPOSITION 10. *A function $\Phi \in L^2(W)$ belongs to $\mathscr{W}_\infty$ if and only if $R(P_{R(W)}\Phi^*) \subseteq H$ σ-a.e.*

Proof. A function $\Phi \in L^2(W)$ belongs to $\mathscr{W}_\infty$ if and only if it is (matrix) orthogonal to $\mathscr{V}_g$ or, equivalently, if and only if $U\Phi$ is (matrix) orthogonal to $U\mathscr{V}_g$ for all $g \in G$. Hence, by Lemma 9, Φ belongs to $\mathscr{W}_\infty$ if and only if

$$\int \langle g, \gamma\rangle AW^{\#}(\gamma) W(\gamma) \Phi^*(\gamma)\sigma(d\gamma) = \int \langle g, \gamma\rangle AP_{R(W(\gamma))}\Phi^*(\gamma)\sigma(d\gamma) = 0$$

for all $g \in G$ and all $A \in M_q$ such that $\mathrm{Ker}(A) \supseteq H$. This in turn is equivalent to the fact that $AP_{R(W)}\Phi^* = 0$ σ-a.e. for all $A \in M_q$ such that $\mathrm{Ker}(A) \supseteq H$, and the result of the proposition follows.

4. Although we do not have an effective recipe for determining the space H, the result of the preceding proposition enables us to give a new proof of Theorem 5.3 in [4].

LEMMA 11. *If $\int W^{\#} d\sigma$ exists and for some subspace K of C^q, $R(W) = K$ σ-a.e., then X is J_0-regular.*

Proof. If $\int W^{\#} d\sigma$ exists, then $H_1 = \{0\}$, and if $R(W) = K$, then $H_2 = K^\perp$. Thus $H = K^\perp$. If Φ belongs to $\mathscr{W}_\infty$, then $R(P_{R(W)}\Phi^*) \subseteq H$, by Proposition 10. On the other hand, $R(P_{R(W)}\Phi^*) \subseteq R(P_{R(W)}) = H^\perp$. It follows that $P_{R(W)}\Phi^* = 0$. This implies $\Phi = 0$ in $L^2(W)$.

LEMMA 12. *If the process X is J_0-regular, then $\int W^{\#} d\sigma$ exists and $R(W) = H^\perp$ σ-a.e.*

Proof. The function $\Phi := P_H$ belongs to $\mathscr{W}_\infty$ (cf. Lemma 8 and Proposition 10). Since, by assumption, $\mathscr{W}_\infty = \{0\}$, we get $\int P_H W P_H d\sigma = 0$ or $P_H W^{1/2} = 0$ σ-a.e., and hence $R(W) \subseteq H^\perp$. But since $R(W) \supseteq H_2^\perp \supseteq H^\perp$, we obtain $R(W) = H^{\perp}$. Finally, we have $P_{H^\perp} W^{\#} P_{H^\perp} = W^{\#}$ and from Lemma 7 the existence of $\int P_{H^\perp} W^{\#} P_{H^\perp} d\sigma$ follows.

Combining Lemmas 11 and 12, we obtain Theorem 5.3 and, as an immediate consequence of Theorem 5.3, Theorem 5.2 in [4].

5. Since the process X is J_0-singular if and only if it is not minimal (cf. [3], Definition 4.1, for the definition of minimality) and since Theorem 4.6 in [3] contains several conditions equivalent to the minimality of X, it is trivial to formulate criteria for J_0-singularity. For example, from Theorem 4.6 (c) in [3] it follows that X is J_0-singular if and only if for all orthoprojectors $P \in M_q$ such that $\operatorname{Ker}(P) \supseteq \operatorname{Ker}(W)$ σ-a.e. the integral $\int PW^{\#} P d\sigma$ is equal to zero or does not exist. It is not hard to see that the following proposition contains an equivalent statement.

PROPOSITION 13. *A process X is J_0-singular if and only if for all $u \in H_2^{\perp}$, $u \neq 0$, the integral $\int (W^{\#} u, u) d\sigma$ does not exist.*

Unfortunately, it seems to be rather difficult to use the preceding criterion in practice. Thus it makes sense to search for necessary or sufficient conditions of J_0-singularity, which are easier to handle. We obtain some results in this direction if we apply Matveev's method [5] to our situation. To formulate them let us introduce some notation.

If $W =: (w_{jk})_{j,k=1}^{q}$ is the spectral density matrix of X, we set

$$W_n := (w_{jk})_{j,k=1}^{n}, \quad n = 1, \dots, q, \ W_0 = 0.$$

Moreover, we use the convention $0^{-1} = \infty$.

PROPOSITION 14. *If X is J_0-singular, then*

$$\int \left[w_{qq} - (w_{q1} \dots w_{q,q-1}) W_{q-1}^{\#} \begin{pmatrix} w_{1q} \\ \vdots \\ w_{q-1,q} \end{pmatrix} \right]^{-1} d\sigma = \infty. \tag{3}$$

If

$$\int \left[w_{jj} - (w_{j1} \dots w_{j,j-1}) W_{j-1}^{\#} \begin{pmatrix} w_{1j} \\ \vdots \\ w_{j-1,j} \end{pmatrix} \right]^{-1} d\sigma = \infty, \tag{4}$$

$j = 2, \dots, q$ and $\int w_{11}^{-1} d\sigma = \infty$, then X is J_0-singular.

Proof. Use the formula for the spectral measure of the orthogonal projection of a stationary process onto a stationary cross-correlated process (cf. Theorem 1.8 in [6]), and the arguments in the proofs of Theorems 1 and 2 in [5]. We omit the details.

COROLLARY. *Assume that* $\det(W) > 0$ *σ-a.e. If X is J_0-singular, then*

$$\int \frac{\det(W_{q-1})}{\det(W_q)} d\sigma = \infty.$$

If

$$\int \frac{\det(W_{j-1})}{\det(W_j)}\, d\sigma = \infty,$$

$j = 2, \ldots, q$, *and* $\int w_{11}^{-1}\, d\sigma = \infty$, *then* X *is* J_0*-singular.*

Proof. The results are immediate consequences of Proposition 14 and the well-known relation

$$\det(W_j) = \det(W_{j-1}) \left[w_{jj} - (w_{j1} \ldots w_{j,j-1})\, W_{j-1}^{-1} \begin{pmatrix} w_{1j} \\ \vdots \\ w_{j-1,j} \end{pmatrix} \right].$$

With the aid of Proposition 13 it is easy to construct an example that shows that (3) is not sufficient for J_0-singularity. On the other hand, since $\int w_{11}^{-1}\, d\sigma = \infty$ implies that the first component (or, by the change of indices, at least one component) of the process X is J_0-singular, it follows that condition (4) is not necessary for J_0-singularity. In fact, the 2-variate process

$$X_n = \begin{pmatrix} X_n^{(1)} \\ X_n^{(2)} \end{pmatrix}, \quad X_n^{(1)} := e_n, \quad X_n^{(2)} := e_{n+1},\ n \in Z,$$

where $(e_n)_{n \in Z}$ is an orthonormal system, is J_0-singular, but both its components are J_0-regular.

6. We conclude our paper with the remark on processes of rank 1. A q-variate stationary process is called a *process of rank* 1 if its spectral measure is absolutely continuous and its spectral density matrix has rank 1 σ-a.e.

PROPOSITION 15 (cf. [2], Corollary 2.5.5). *A process of rank* 1 *is either* J_0*-regular or* J_0*-singular.*

Proof. Let the spectral density matrix W of the process X have rank 1. If the range of W is not constant, then $H_2 = C^q$ and, by Proposition 13, X is J_0-singular. If the range of W is constant, then H_2 is a $(q-1)$-dimensional subspace of C^q. Let Q be the orthoprojector onto $H_2^{\perp}$. If $\int QW^{\#} Q d\sigma$ does not exist, then again Proposition 13 yields the J_0-singularity of X. Otherwise, X is J_0-regular according to $QW^{\#} Q = W^{\#}$ and Lemma 11.

REFERENCES

[1] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Vol. I. *Elementary Theory*, Academic Press, New York 1983.

[2] A. Makagon and H. Salehi, *Notes on infinite dimensional stationary sequences*, in: *Probability Theory on Vector Spaces. IV*, S. Cambanis and A. Weron (Eds.), Lecture Notes in Math. 1391 (1989), pp. 200–238.

[3] A. Makagon and A. Weron, *q-variate minimal stationary processes*, Studia Math. 59 (1976), pp. 41–52.
[4] — *Wold–Cramér concordance theorems for interpolation of q-variate stationary processes over locally compact abelian groups*, J. Multivariate Anal. 6 (1976), pp. 123–137.
[5] R. F. Matveev, *On singular multidimensional stationary processes* (in Russian), Teor. Veroyatnost. i Primenen. 5 (1960), pp. 38–44.
[6] M. Rosenberg, *Mutual subordination of multivariate stationary processes over any locally compact group*, Z. Wahrsch. Verw. Gebiete 12 (1969), pp. 333–343.

L. Klotz
Fakultät für Mathematik/Informatik
Universität
04109 Leipzig, Germany

F. Schmidt
Institut für Mathematische Stochastik
Fachrichtung Mathematik
Technische Universität
01062 Dresden, Germany

Received on 11.12.1997